

The Cauchy Singular Integral Operator on Weighted Variable Lebesgue Spaces

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Abstract. Let $p : \mathbb{R} \rightarrow (1, \infty)$ be a globally log-Hölder continuous variable exponent and $w : \mathbb{R} \rightarrow [0, \infty]$ be a weight. We prove that the Cauchy singular integral operator S is bounded on the weighted variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}, w) = \{f : fw \in L^{p(\cdot)}(\mathbb{R})\}$ if and only if the weight w satisfies

$$\sup_{-\infty < a < b < \infty} \frac{1}{b-a} \|w\chi_{(a,b)}\|_{p(\cdot)} \|w^{-1}\chi_{(a,b)}\|_{p'(\cdot)} < \infty \quad (1/p(x) + 1/p'(x) = 1).$$

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1. Introduction

Let $p : \mathbb{R} \rightarrow [1, \infty]$ be a measurable a.e. finite function. By $L^{p(\cdot)}(\mathbb{R})$ we denote the set of all complex-valued functions f on \mathbb{R} such that

$$I_{p(\cdot)}(f/\lambda) := \int_{\mathbb{R}} |f(x)/\lambda|^{p(x)} dx < \infty$$

for some $\lambda > 0$. This set becomes a Banach space when equipped with the norm

$$\|f\|_{p(\cdot)} := \inf \{ \lambda > 0 : I_{p(\cdot)}(f/\lambda) \leq 1 \}.$$

It is easy to see that if p is constant, then $L^{p(\cdot)}(\mathbb{R})$ is nothing but the standard Lebesgue space $L^p(\mathbb{R})$. The space $L^{p(\cdot)}(\mathbb{R})$ is referred to as a *variable Lebesgue space*.

A measurable function $w : \mathbb{R} \rightarrow [0, \infty]$ is referred to as a *weight* whenever $0 < w(x) < \infty$ a.e. on \mathbb{R} . Given a variable exponent $p : \mathbb{R} \rightarrow [1, \infty]$ and a weight

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$w : \mathbb{R} \rightarrow [0, \infty]$, we define the weighted variable exponent space $L^{p(\cdot)}(\mathbb{R}, w)$ as the space of all measurable complex-valued functions f such that $fw \in L^{p(\cdot)}(\mathbb{R})$. The norm on this space is naturally defined by

$$\|f\|_{p(\cdot), w} := \|fw\|_{p(\cdot)}.$$

Given $f \in L^1_{\text{loc}}(\mathbb{R})$, the Hardy-Littlewood maximal operator is defined by

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy$$

where the supremum is taken over all intervals $Q \subset \mathbb{R}$ containing x . The Cauchy singular integral operator S is defined for $f \in L^1_{\text{loc}}(\mathbb{R})$ by

$$(Sf)(x) := \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(\tau)}{\tau - x} d\tau \quad (x \in \mathbb{R}),$$

where the integral is understood in the principal value sense.

Following [4, Section 2] or [5, Section 4.1], one says that $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is locally log-Hölder continuous if there exists $c_1 > 0$ such that

$$|\alpha(x) - \alpha(y)| \leq \frac{c_1}{\log(e + 1/|x - y|)}$$

for all $x, y \in \mathbb{R}$. Further, α is said to satisfy the log-Hölder decay condition if there exist $\alpha_\infty \in \mathbb{R}$ and a constant $c_2 > 0$ such that

$$|\alpha(x) - \alpha_\infty| \leq \frac{c_2}{\log(e + |x|)}$$

for all $x \in \mathbb{R}$. One says that α is globally log-Hölder continuous on \mathbb{R} if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. Put

$$p_- := \operatorname{ess\,inf}_{x \in \mathbb{R}} p(x), \quad \operatorname{ess\,sup}_{x \in \mathbb{R}} p(x) =: p_+.$$

As usual, we use the convention $1/\infty := 0$ and denote by $\mathcal{P}^{\log}(\mathbb{R})$ the set of all variable exponents such that $1/p$ is globally log-Hölder continuous. If $p \in \mathcal{P}^{\log}(\mathbb{R})$, then the limit

$$\frac{1}{p(\infty)} := \lim_{|x| \rightarrow \infty} \frac{1}{p(x)}$$

exists. If $p_+ < \infty$, then $p \in \mathcal{P}^{\log}(\mathbb{R})$ if and only if p is globally log-Hölder continuous.

By [5, Theorem 4.3.8], if $p \in \mathcal{P}^{\log}(\mathbb{R})$ with $p_- > 1$, then the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R})$. Notice, however, that the condition $p \in \mathcal{P}^{\log}(\mathbb{R})$ is not necessary, there are even discontinuous exponents p such that M is bounded on $L^{p(\cdot)}(\mathbb{R})$. Corresponding examples were first constructed by Lerner and they are contained in [5, Section 5.1].

In this paper we will mainly suppose that

$$1 < p_-, \quad p_+ < \infty. \tag{1.1}$$

Under these conditions, the space $L^{p(\cdot)}(\mathbb{R})$ is separable and reflexive, and its Banach space dual $[L^{p(\cdot)}(\mathbb{R})]^*$ is isomorphic to $L^{p'(\cdot)}(\mathbb{R})$, where

$$1/p(x) + 1/p'(x) = 1 \quad (x \in \mathbb{R})$$

(see [5, Chap. 3]). If, in addition, $w\chi_E \in L^{p(\cdot)}(\mathbb{R})$ and $\chi_E/w \in L^{p'(\cdot)}(\mathbb{R})$ for any measurable set $E \subset \mathbb{R}$ of finite measure, then $L^{p(\cdot)}(\mathbb{R}, w)$ is a Banach function space and $[L^{p(\cdot)}(\mathbb{R}, w)]^* = L^{p(\cdot)}(\mathbb{R}, w^{-1})$. Here and in what follows, χ_E denotes the characteristic function of the set E .

Probably, one of the simplest weights is the following power weight

$$w(x) := |x - i|^{\lambda_\infty} \prod_{j=1}^m |x - x_j|^{\lambda_j}, \quad (1.2)$$

where $-\infty < x_1 < \dots < x_m < +\infty$ and $\lambda_1, \dots, \lambda_m, \lambda_\infty \in \mathbb{R}$. Kokilashvili, Paatashvili, and Samko studied the boundedness of the operators M and S on $L^{p(\cdot)}(\mathbb{R}, w)$ with power weights (1.2). From [11, Theorem A] and [14, Theorem B] one can extract the following result.

Theorem 1.1. *Let $p \in \mathcal{P}^{\log}(\mathbb{R})$ satisfy (1.1) and w be a power weight (1.2).*

- (a) **(Kokilashvili, Samko).** *Suppose, in addition, that p is constant outside an interval containing x_1, \dots, x_m . Then the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}, w)$ if and only if*

$$0 < \frac{1}{p(x_j)} + \lambda_j < 1 \text{ for } j \in \{1, \dots, m\}, \quad 0 < \frac{1}{p(\infty)} + \lambda_\infty + \sum_{j=1}^m \lambda_j < 1. \quad (1.3)$$

- (b) **(Kokilashvili, Paatashvili, Samko).** *The Cauchy singular integral operator S is bounded on $L^{p(\cdot)}(\mathbb{R}, w)$ if and only if (1.3) is fulfilled.*

Further, the sufficiency portion of this result was extended in [12, 13] to radial oscillating weights of the form $\prod_{j=1}^m \omega_j(|x - x_j|)$, where $\omega_j(t)$ are continuous functions for $t > 0$ that may oscillate near zero and whose Matuszewska-Orlicz indices can be different. Notice that the Matuszewska-Orlicz indices of $\omega_j(t) = t^{\lambda_j}$ are both equal to λ_j .

Very recently, Cruz-Uribe, Diening, and Hästö [5, Theorem 1.3] generalized part (a) of Theorem 1.1 to the case of general weights. To formulate their result, we will introduce the following generalization of the classical Muckenhoupt condition (written in the symmetric form). We say that a weight $w : \mathbb{R} \rightarrow [0, \infty]$ belongs to the class $\mathcal{A}_{p(\cdot)}(\mathbb{R})$ if

$$\sup_{-\infty < a < b < \infty} \frac{1}{b-a} \|w\chi_{(a,b)}\|_{p(\cdot)} \|w^{-1}\chi_{(a,b)}\|_{p'(\cdot)} < \infty.$$

This condition goes back to Bereznoi [3] (in the more general setting of Banach function spaces), it was studied by the first author [8] (in the case of Banach function spaces defined on Carleson curves) and Kopalani [15].

Theorem 1.2 (Cruz-Uribe, Diening, Hästö). *Let $p \in \mathcal{P}^{\log}(\mathbb{R})$ satisfy (1.1) and $w : \mathbb{R} \rightarrow [0, \infty]$ be a weight. The Hardy-Littlewood maximal operator M is bounded on the weighted variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}, w)$ if and only if $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R})$.*

The aim of this paper is to generalize part (b) of Theorem 1.1 to the case of general weights. We will prove the following.

Theorem 1.3 (Main result). *Let $p \in \mathcal{P}^{\log}(\mathbb{R})$ satisfy (1.1) and $w : \mathbb{R} \rightarrow [0, \infty]$ be a weight. The Cauchy singular integral operator S is bounded on the weighted variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}, w)$ if and only if $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R})$.*

From this theorem, by using standard techniques, we derive also the following.

Theorem 1.4. *Let $p \in \mathcal{P}^{\log}(\mathbb{R})$ satisfy (1.1) and $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R})$. Then $S^2 = I$ on the space $L^{p(\cdot)}(\mathbb{R}, w)$ and $S^* = S$ on the space $L^{p'(\cdot)}(\mathbb{R}, w^{-1})$.*

The paper is organized as follows. In Section 2 we collect necessary facts on Banach function spaces $X(\mathbb{R})$ in the sense of Luxemburg and discuss weighted Banach function spaces $X(\mathbb{R}, w) = \{f : fw \in X(\mathbb{R})\}$. A special attention is paid to conditions implying that $X(\mathbb{R}, w)$ is a Banach function space itself, to separability and reflexivity of $X(\mathbb{R}, w)$, and to density of smooth compactly supported functions in $X(\mathbb{R}, w)$ and in its dual space $X'(\mathbb{R}, w^{-1})$. In Section 3.2 we prepare the proof of a sufficient condition for the boundedness of the operator S and formulate two key estimates by Lerner [16] and Álvarez and Pérez [1]. On the basis of these results, in Section 3.3 we prove that if $X(\mathbb{R})$ is a separable Banach function space and the Hardy-Littlewood maximal function is bounded on the weighted Banach function spaces $X(\mathbb{R}, w)$ and $X'(\mathbb{R}, w^{-1})$, then S is bounded on $X(\mathbb{R}, w)$ and $S^2 = I$. Moreover, if $X(\mathbb{R})$ is reflexive, then S^* coincides with S on $X'(\mathbb{R}, w^{-1})$. In Section 3.4 we prove that if S is bounded on the weighted Banach function spaces $X(\mathbb{R}, w)$, then

$$\sup_{-\infty < a < b < \infty} \frac{1}{b-a} \|w\chi_{(a,b)}\|_{X(\mathbb{R})} \|w^{-1}\chi_{(a,b)}\|_{X'(\mathbb{R})} < \infty$$

where $X'(\mathbb{R})$ is the associate space for $X(\mathbb{R})$. Finally, in Section 3.5 we explain that Theorems 1.3 and 1.4 follow from results of Sections 3.3–3.4 and Theorem 1.2 because $L^{p(\cdot)}(\mathbb{R})$ is a Banach function space, which is separable and reflexive whenever p satisfies (1.1).

2. Weighted Banach function spaces

2.1. Banach function spaces

The set of all Lebesgue measurable complex-valued functions on \mathbb{R} is denoted by \mathcal{M} . Let \mathcal{M}^+ be the subset of functions in \mathcal{M} whose values lie in $[0, \infty]$. The characteristic function of a measurable set $E \subset \mathbb{R}$ is denoted by χ_E and the Lebesgue measure of E is denoted by $|E|$.

Definition 2.1 ([2, Chap. 1, Definition 1.1]). A mapping $\rho : \mathcal{M}^+ \rightarrow [0, \infty]$ is called a *Banach function norm* if, for all functions f, g, f_n ($n \in \mathbb{N}$) in \mathcal{M}^+ , for all constants $a \geq 0$, and for all measurable subsets E of \mathbb{R} , the following properties hold:

- (A1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f)$, $\rho(f + g) \leq \rho(f) + \rho(g)$,
- (A2) $0 \leq g \leq f$ a.e. $\Rightarrow \rho(g) \leq \rho(f)$ (the lattice property),
- (A3) $0 \leq f_n \uparrow f$ a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$ (the Fatou property),
- (A4) $|E| < \infty \Rightarrow \rho(\chi_E) < \infty$,
- (A5) $|E| < \infty \Rightarrow \int_E f(x) dx \leq C_E \rho(f)$

with $C_E \in (0, \infty)$ which may depend on E and ρ but is independent of f .

When functions differing only on a set of measure zero are identified, the set $X(\mathbb{R})$ of all functions $f \in \mathcal{M}$ for which $\rho(|f|) < \infty$ is called a *Banach function space*. For each $f \in X(\mathbb{R})$, the norm of f is defined by

$$\|f\|_{X(\mathbb{R})} := \rho(|f|).$$

The set $X(\mathbb{R})$ under the natural linear space operations and under this norm becomes a Banach space (see [2, Chap. 1, Theorems 1.4 and 1.6]).

If ρ is a Banach function norm, its associate norm ρ' is defined on \mathcal{M}^+ by

$$\rho'(g) := \sup \left\{ \int_{\mathbb{R}} f(x)g(x) dx : f \in \mathcal{M}^+, \rho(f) \leq 1 \right\}, \quad g \in \mathcal{M}^+.$$

It is a Banach function norm itself [2, Chap. 1, Theorem 2.2]. The Banach function space $X'(\mathbb{R})$ determined by the Banach function norm ρ' is called the *associate space* (*Köthe dual*) of $X(\mathbb{R})$. The associate space $X'(\mathbb{R})$ is a subspace of the dual space $[X(\mathbb{R})]^*$. The construction of the associate space implies the following Hölder inequality for Banach function spaces.

Lemma 2.2 ([2, Chap. 1, Theorem 2.4]). *Let $X(\mathbb{R})$ be a Banach function space and $X'(\mathbb{R})$ be its associate space. If $f \in X(\mathbb{R})$ and $g \in X'(\mathbb{R})$, then fg is integrable and*

$$\|fg\|_{L^1(\mathbb{R})} \leq \|f\|_{X(\mathbb{R})} \|g\|_{X'(\mathbb{R})}.$$

The next result provides a useful converse to the integrability assertion of Lemma 2.2.

Lemma 2.3 ([2, Chap. 1, Lemma 2.6]). *Let $X(\mathbb{R})$ be a Banach function space. In order that a measurable function g belong to the associate space $X'(\mathbb{R})$, it is necessary and sufficient that fg be integrable for every f in $X(\mathbb{R})$.*

2.2. Weighted Banach function spaces

Let $X(\mathbb{R})$ be a Banach function space generated by a Banach function norm ρ . We say that $f \in X_{\text{loc}}(\mathbb{R})$ if $f\chi_E \in X(\mathbb{R})$ for any measurable set $E \subset \mathbb{R}$ of finite measure. A measurable function $w : \mathbb{R} \rightarrow [0, \infty]$ is referred to as a *weight* if

$0 < w(x) < \infty$ a.e. on \mathbb{R} . Define the mapping $\rho_w : \mathcal{M}^+ \rightarrow [0, \infty]$ and the set $X(\mathbb{R}, w)$ by

$$\rho_w(f) := \rho(fw) \quad (f \in \mathcal{M}^+), \quad X(\mathbb{R}, w) := \{f \in \mathcal{M}^+ : fw \in X(\mathbb{R})\}.$$

Lemma 2.4. *Let $X(\mathbb{R})$ be a Banach function space generated by a Banach function norm ρ , let $X'(\mathbb{R})$ be its associate space, and let $w : \mathbb{R} \rightarrow [0, \infty]$ be a weight.*

- (a) *The mapping ρ_w satisfies Axioms (A1)-(A3) in Definition 2.1 and $X(\mathbb{R}, w)$ is a linear normed space with respect to the norm*

$$\|f\|_{X(\mathbb{R}, w)} := \rho_w(|f|) = \rho(|fw|) = \|fw\|_{X(\mathbb{R})}.$$

- (b) *If $w \in X_{\text{loc}}(\mathbb{R})$ and $1/w \in X'_{\text{loc}}(\mathbb{R})$, then ρ_w is a Banach function norm and $X(\mathbb{R}, w)$ is a Banach function space generated by ρ_w .*
(c) *If $w \in X_{\text{loc}}(\mathbb{R})$ and $1/w \in X'_{\text{loc}}(\mathbb{R})$, then $X'(\mathbb{R}, w^{-1})$ is the associate space for the Banach function space $X(\mathbb{R}, w)$.*

Proof. The proof is analogous to that one of [8, Lemma 2.5].

Part (a) follows from Axioms (A1)-(A3) for the Banach function norm ρ and the fact that $0 < w(x) < \infty$ almost everywhere on \mathbb{R} .

(b) If $w \in X_{\text{loc}}(\mathbb{R})$, then $w\chi_E \in X(\mathbb{R})$ for every measurable set $E \subset \mathbb{R}$ of finite measure. Therefore $\rho_w(\chi_E) = \rho(w\chi_E) < \infty$. Then ρ_w satisfies Axiom (A4).

Since $1/w \in X'_{\text{loc}}(\mathbb{R})$, we have $C_E := \rho'(\chi_E/w) < \infty$ for every measurable set $E \subset \mathbb{R}$ of finite measure. On the other hand, by Axiom (A2), for $f \in \mathcal{M}^+$ we have $\rho(fw\chi_E) \leq \rho(fw) = \rho_w(f)$. By Hölder's inequality for ρ (Lemma 2.2), we obtain

$$\int_E f(x) dx = \int_{\mathbb{R}} f(x)w(x)\chi_E(x) \cdot \frac{\chi_E(x)}{w(x)} dx \leq \rho(fw\chi_E)\rho'(\chi_E/w) \leq C_E\rho_w(f).$$

Thus ρ_w satisfies Axiom (A5), that is, $X(\mathbb{R}, w)$ is a Banach function space. Part (b) is proved.

- (c) For $g \in \mathcal{M}^+$, we have

$$\begin{aligned} (\rho_w)'(g) &= \sup \left\{ \int_{\mathbb{R}} f(x)g(x) dx : f \in \mathcal{M}^+, \rho_w(f) \leq 1 \right\} \\ &= \sup \left\{ \int_{\mathbb{R}} (f(x)w(x)) \left(\frac{g(x)}{w(x)} \right) dx : f \in \mathcal{M}^+, \rho(fw) \leq 1 \right\} \\ &= \sup \left\{ \int_{\mathbb{R}} h(x) \left(\frac{g(x)}{w(x)} \right) dx : h \in \mathcal{M}^+, \rho(h) \leq 1 \right\} \\ &= \rho'(g/w). \end{aligned}$$

Hence $(X(\mathbb{R}, w))' = X'(\mathbb{R}, w^{-1})$. \square

From Lemma 2.4 and the Lorentz-Luxemburg theorem (see e.g. [2, Chap. 1, Theorem 2.7]) we obtain the following.

Lemma 2.5. *Let $X(\mathbb{R})$ be a Banach function space and $w : \mathbb{R} \rightarrow [0, \infty]$ be a weight such that $w \in X_{\text{loc}}(\mathbb{R})$ and $1/w \in X'_{\text{loc}}(\mathbb{R})$. Then*

$$\|f\|_{X(\mathbb{R}, w)} = \sup \left\{ \int_{\mathbb{R}} |f(x)g(x)| dx : g \in X'(\mathbb{R}, w^{-1}), \|g\|_{X'(\mathbb{R}, w^{-1})} \leq 1 \right\} \quad (2.1)$$

for all $f \in X(\mathbb{R}, w)$ and

$$\|g\|_{X'(\mathbb{R}, w^{-1})} = \sup \left\{ \int_{\mathbb{R}} |f(x)g(x)| dx : f \in X(\mathbb{R}, w), \|f\|_{X(\mathbb{R}, w)} \leq 1 \right\} \quad (2.2)$$

for all $g \in X'(\mathbb{R}, w^{-1})$.

2.3. Reflexivity of weighted Banach function spaces

A function f in a Banach function space $X(\mathbb{R})$ is said to have *absolutely continuous norm* in $X(\mathbb{R})$ if $\|f\chi_{E_n}\|_{X(\mathbb{R})} \rightarrow 0$ for every sequence $\{E_n\}_{n=1}^{\infty}$ of measurable sets on \mathbb{R} satisfying $\chi_{E_n} \rightarrow 0$ a.e. on \mathbb{R} as $n \rightarrow \infty$. If all functions $f \in X(\mathbb{R})$ have this property, then the space $X(\mathbb{R})$ itself is said to have *absolutely continuous norm* (see [2, Chap. 1, Section 3]).

Lemma 2.6 ([2, Chap. 1, Lemma 3.4]). *Let $X(\mathbb{R})$ be a Banach function space. If $f \in X(\mathbb{R})$ has absolutely continuous norm, then to each $\varepsilon > 0$ there corresponds $\delta > 0$ such that $|E| < \delta$ implies $\|f\chi_E\|_{X(\mathbb{R})} < \varepsilon$.*

Lemma 2.7. *Let $X(\mathbb{R})$ be a Banach function space and $w : \mathbb{R} \rightarrow [0, \infty]$ be a weight such that $w \in X_{\text{loc}}(\mathbb{R})$ and $1/w \in X'_{\text{loc}}(\mathbb{R})$. If $X(\mathbb{R})$ has absolutely continuous norm, then $X(\mathbb{R}, w)$ has absolutely continuous norm too.*

Proof. The proof is a literal repetition of that one of [8, Proposition 2.6]. By Lemma 2.4(b), $X(\mathbb{R}, w)$ is a Banach function space. If $f \in X(\mathbb{R}, w)$, then $fw \in X(\mathbb{R})$ has absolutely continuous norm in $X(\mathbb{R})$. Therefore,

$$\|f\chi_{E_n}\|_{X(\mathbb{R}, w)} = \|fw\chi_{E_n}\|_{X(\mathbb{R})} \rightarrow 0$$

for every sequence $\{E_n\}_{n=1}^{\infty}$ of measurable sets on \mathbb{R} satisfying $\chi_{E_n} \rightarrow 0$ a.e. on \mathbb{R} as $n \rightarrow \infty$. Thus, $f \in X(\mathbb{R}, w)$ has absolutely continuous norm in $X(\mathbb{R}, w)$. \square

From Lemma 2.4 and [2, Chap. 1, Corollaries 4.3, 4.4] we obtain the following.

Lemma 2.8. *Let $X(\mathbb{R})$ be a Banach function space and $w : \mathbb{R} \rightarrow [0, \infty]$ be a weight such that $w \in X_{\text{loc}}(\mathbb{R})$ and $1/w \in X'_{\text{loc}}(\mathbb{R})$.*

- (a) *The Banach space dual $[X(\mathbb{R}, w)]^*$ of the weighted Banach function space $X(\mathbb{R}, w)$ is isometrically isomorphic to the associate space $X'(\mathbb{R}, w^{-1})$ if and only if $X(\mathbb{R}, w)$ has absolutely continuous norm. If this is the case, then the general form of a linear functional on $X(\mathbb{R}, w)$ is given by*

$$G(f) := \int_{\mathbb{R}} f(x)\overline{g(x)} dx \quad \text{for } g \in X'(\mathbb{R}, w^{-1})$$

and $\|G\|_{[X(\mathbb{R}, w)]^} = \|g\|_{X'(\mathbb{R}, w^{-1})}$.*

- (b) *The weighted Banach function space $X(\mathbb{R}, w)$ is reflexive if and only if both $X(\mathbb{R}, w)$ and $X'(\mathbb{R}, w^{-1})$ have absolutely continuous norm.*

Corollary 2.9. *Let $X(\mathbb{R})$ be a Banach function space and $w : \mathbb{R} \rightarrow [0, \infty]$ be a weight such that $w \in X_{\text{loc}}(\mathbb{R})$ and $1/w \in X'_{\text{loc}}(\mathbb{R})$. If $X(\mathbb{R})$ is reflexive, then $X(\mathbb{R}, w)$ is reflexive.*

Proof. The proof is a literal repetition of that one of [8, Corollary 2.8]. If $X(\mathbb{R})$ is reflexive, then, by [2, Chap. 1, Corollary 4.4], both $X(\mathbb{R})$ and $X'(\mathbb{R})$ have absolutely continuous norm. In that case, due to Lemma 2.7, both $X(\mathbb{R}, w)$ and $X'(\mathbb{R}, w^{-1})$ have absolutely continuous norm. By Lemma 2.8(b), $X(\mathbb{R}, w)$ is reflexive. \square

2.4. Density of smooth compactly supported functions

For a subset Y of $L^\infty(\mathbb{R})$, let Y_0 denote the set of all compactly supported functions in Y .

Lemma 2.10. *Let $X(\mathbb{R})$ be a Banach function space.*

- (a) $L_0^\infty(\mathbb{R}) \subset X(\mathbb{R})$.
- (b) *If $X(\mathbb{R})$ has absolutely continuous norm, then $L_0^\infty(\mathbb{R})$, $C_0(\mathbb{R})$, and $C_0^\infty(\mathbb{R})$ are dense in $X(\mathbb{R})$.*

Proof. Part (a) follows from the definition of a Banach function space.

(b) From [2, Chap. 1, Proposition 3.10 and Theorem 3.11] it follows that $L_0^\infty(\mathbb{R})$ is dense in $X(\mathbb{R})$.

Let us show that each function $u \in L_0^\infty(\mathbb{R})$ can be approximated by a function from $C_0(\mathbb{R})$ in the norm of $X(\mathbb{R})$. We have $\text{supp } u \subset Q$ and $|u(x)| \leq a$ for almost all $x \in \mathbb{R}$, where Q is some finite closed segment and $a > 0$. By Axiom (A4), $\chi_Q \in X(\mathbb{R})$ and χ_Q has absolutely continuous norm by the hypothesis. From Lemma 2.6 it follows that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|E| < \delta$ implies that $\|\chi_Q \chi_E\|_{X(\mathbb{R})} < \varepsilon$. By Luzin's theorem, for such a $\delta > 0$ there is a continuous function v supported in Q such that $|v(x)| \leq a$ and the measure of the set $\tilde{Q} := \{x \in Q : u(x) \neq v(x)\}$ is less than δ . Then

$$|u(x) - v(x)| \leq 2a\chi_{\tilde{Q}}(x) \quad (x \in \mathbb{R}).$$

Therefore, by Axiom (A2),

$$\|u - v\|_{X(\mathbb{R})} \leq 2a\|\chi_Q \chi_{\tilde{Q}}\|_{X(\mathbb{R})} < 2a\varepsilon.$$

Hence, each function $u \in L_0^\infty(\mathbb{R})$ can be approximated by a function from $C_0(\mathbb{R})$ in the norm of $X(\mathbb{R})$. Thus, $C_0(\mathbb{R})$ is dense in $X(\mathbb{R})$.

Now let us prove that each function $v \in C_0(\mathbb{R})$ can be approximated by a function from $C_0^\infty(\mathbb{R})$ in the norm of $X(\mathbb{R})$. Let $a \in C_0^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} a(x)dx = 1$. Consider

$$v_t(x) = \frac{1}{t} \int_{\mathbb{R}} a\left(\frac{y}{t}\right) v(x - y) dy \quad (t > 0).$$

It is easy to see that $v_t \in C_0^\infty(\mathbb{R})$. Fix an interval Q containing the supports of v and v_t . Then for every $\varepsilon > 0$ there is a $t > 0$ such that $|v_t(x) - v(x)| < \varepsilon$ for all $x \in Q$. Hence,

$$\|v_t - v\|_{X(\mathbb{R})} = \|(v_t - v)\chi_Q\|_{X(\mathbb{R})} < \varepsilon\|\chi_Q\|_{X(\mathbb{R})},$$

that is, $v \in C_0(\mathbb{R})$ can be approximated by a function from $C_0^\infty(\mathbb{R})$ in the norm of $X(\mathbb{R})$. Thus, $C_0^\infty(\mathbb{R})$ is dense in $X(\mathbb{R})$. \square

From [2, Chap. 1, Corollary 5.6] one can extract the following.

Lemma 2.11. *A Banach function space $X(\mathbb{R})$ is separable if and only if it has absolutely continuous norm.*

Gathering the results mentioned above, we arrive at the next result.

Lemma 2.12. *Let $X(\mathbb{R})$ be a Banach function space and $w : \mathbb{R} \rightarrow [0, \infty]$ be a weight such that $w \in X_{\text{loc}}(\mathbb{R})$ and $1/w \in X'_{\text{loc}}(\mathbb{R})$.*

- (a) *If $X(\mathbb{R})$ is separable, then $L_0^\infty(\mathbb{R})$, $C_0(\mathbb{R})$, and $C_0^\infty(\mathbb{R})$ are dense in the weighted Banach function space $X(\mathbb{R}, w)$.*
- (b) *If $X(\mathbb{R})$ is reflexive, then $L_0^\infty(\mathbb{R})$, $C_0(\mathbb{R})$, and $C_0^\infty(\mathbb{R})$ are dense in the weighted Banach function spaces $X(\mathbb{R}, w)$ and $X'(\mathbb{R}, w^{-1})$.*

Proof. (a) If $X(\mathbb{R})$ is separable, then by Lemma 2.11, $X(\mathbb{R})$ has absolutely continuous norm. Therefore $X(\mathbb{R}, w)$ has absolutely continuous norm too, in view of Lemma 2.7. Hence, from Lemma 2.10(b) we derive that $L_0^\infty(\mathbb{R})$, $C_0(\mathbb{R})$, and $C_0^\infty(\mathbb{R})$ are dense in $X(\mathbb{R}, w)$. Part (a) is proved.

(b) If $X(\mathbb{R})$ is reflexive, then by [2, Chap. 1, Corollary 4.4], both $X(\mathbb{R})$ and $X'(\mathbb{R})$ have absolutely continuous norm. Hence both $X(\mathbb{R}, w)$ and $X'(\mathbb{R}, w^{-1})$ have absolutely continuous norm in view of Lemma 2.7. Thus, from Lemma 2.10(b) we get that $L_0^\infty(\mathbb{R})$, $C_0(\mathbb{R})$, and $C_0^\infty(\mathbb{R})$ are dense in $X(\mathbb{R}, w)$ and in $X'(\mathbb{R}, w^{-1})$. \square

3. Boundedness of the Cauchy singular integral operator on weighted Banach function spaces

3.1. Well-known properties of the Cauchy singular integral operator

One says that a linear operator T from $L^1(\mathbb{R})$ into the space of complex-valued measurable functions on \mathbb{R} is of weak type $(1, 1)$ if for every $\alpha > 0$,

$$|\{x \in \mathbb{R} : |(Tf)(x)| > \alpha\}| \leq \frac{C_K}{\alpha} \|f\|_{L^1(\mathbb{R})}$$

with some absolute constant $C_K > 0$.

The following results are proved in many standard texts on Harmonic Analysis, see e.g. [2, Chap. 3, Theorem 4.9(b)] or [6, pp. 51–52].

Theorem 3.1. (a) **(Kolmogorov).** *The Cauchy singular integral operator S is of weak type $(1, 1)$.*

- (b) **(M. Riesz).** *The Cauchy singular integral operator S is bounded on $L^p(\mathbb{R})$ for every $p \in (1, \infty)$.*

Theorem 3.2. *If $f, g \in L^2(\mathbb{R})$, then*

$$(S^2 f)(x) = f(x) \quad (x \in \mathbb{R}), \quad (3.1)$$

$$\int_{\mathbb{R}} (Sf)(x) \overline{g(x)} dx = \int_{\mathbb{R}} f(x) \overline{(Sg)(x)} dx. \quad (3.2)$$

3.2. Pointwise estimates for sharp maximal functions

For $\delta > 0$ and $f \in L_{\text{loc}}^\delta(\mathbb{R})$, set

$$f_\delta^\#(x) := \sup_{Q \ni x} \inf_{c \in \mathbb{R}} \left(\frac{1}{|Q|} \int_Q |f(y) - c|^\delta dy \right)^{1/\delta}.$$

The non-increasing rearrangement (see, e.g., [2, Chap. 2, Section 1]) of a measurable function f on \mathbb{R} is defined by

$$f^*(t) := \inf \{ \lambda > 0 : |\{x \in \mathbb{R} : |f(x)| > \lambda\}| \leq t \} \quad (0 < t < \infty).$$

For a fixed $\lambda \in (0, 1)$ and a given measurable function f on \mathbb{R} , consider the local sharp maximal function $M_\lambda^\# f$ defined by

$$M_\lambda^\# f(x) := \sup_{Q \ni x} \inf_{c \in \mathbb{R}} ((f - c)\chi_Q)^* (\lambda|Q|).$$

In all above definitions the suprema are taken over all intervals $Q \subset \mathbb{R}$ containing x .

The following result was proved by Lerner [16, Theorem 1] for the case of \mathbb{R}^n .

Theorem 3.3 (Lerner). *For a function $g \in L_{\text{loc}}^1(\mathbb{R})$ and a measurable function φ satisfying*

$$|\{x \in \mathbb{R} : |\varphi(x)| > \alpha\}| < \infty \quad \text{for all } \alpha > 0, \quad (3.3)$$

one has

$$\int_{\mathbb{R}} |\varphi(x)g(x)| dx \leq C_L \int_{\mathbb{R}} M_\lambda^\# \varphi(x) M g(x) dx,$$

where $C_L > 0$ and $\lambda \in (0, 1)$ are some absolute constants.

The sharp maximal functions can be related as follows.

Lemma 3.4 ([9, Proposition 2.3]). *If $\delta > 0, \lambda \in (0, 1)$, and $f \in L_{\text{loc}}^\delta(\mathbb{R})$, then*

$$M_\lambda^\# f(x) \leq (1/\lambda)^{1/\delta} f_\delta^\#(x) \quad (x \in \mathbb{R}).$$

The following estimate was proved in [1, Theorem 2.1] for the case of Calderón-Zygmund singular integral operators with standard kernels in the sense of Coifman and Meyer on \mathbb{R}^n . It is well known that the Cauchy kernel is an archetypical example of a standard kernel (see e.g. [6, p. 99]).

Theorem 3.5 (Álvarez-Pérez). *If $0 < \delta < 1$, then for every $f \in C_0^\infty(\mathbb{R})$,*

$$(Sf)_\delta^\#(x) \leq C_\delta M f(x) \quad (x \in \mathbb{R})$$

where $C_\delta > 0$ is some constant depending only on δ .

3.3. Sufficient condition

The set of all bounded sublinear operators on a Banach function space $Y(\mathbb{R})$ will be denoted by $\tilde{\mathcal{B}}(Y(\mathbb{R}))$ and its subset of all bounded linear operators will be denoted by $\mathcal{B}(Y(\mathbb{R}))$.

Theorem 3.6. *Let $X(\mathbb{R})$ be a separable Banach function space and $w : \mathbb{R} \rightarrow [0, \infty]$ be a weight such that $w \in X_{\text{loc}}(\mathbb{R})$ and $1/w \in X'_{\text{loc}}(\mathbb{R})$. Suppose the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R}, w)$ and on $X'(\mathbb{R}, w^{-1})$. Assume that $0 < \delta < 1$ and T is an operator such that*

- (a) T is of weak type $(1, 1)$;
- (b) T is bounded on some $L^p(\mathbb{R})$ with $p \in (1, \infty)$;
- (c) for each $f \in C_0^\infty(\mathbb{R})$,

$$(Tf)_\delta^\#(x) \leq C_\delta Mf(x) \quad (x \in \mathbb{R})$$

where C_δ is a positive constant depending only on δ .

Then $T \in \mathcal{B}(X(\mathbb{R}, w))$ and

$$\|T\|_{\mathcal{B}(X(\mathbb{R}, w))} \leq (1/\lambda)^\delta C_L \|M\|_{\tilde{\mathcal{B}}(X(\mathbb{R}, w))} \|M\|_{\mathcal{B}(X'(\mathbb{R}, w^{-1}))} C_\delta, \quad (3.4)$$

where $\lambda \in (0, 1)$ and $C_L > 0$ are the constants from Theorem 3.3.

Proof. The idea of the proof is borrowed from [9, Theorem 2.7]. By Lemma 2.4, $X(\mathbb{R}, w)$ is a Banach function space whose associate space is $X'(\mathbb{R}, w^{-1})$. Let $f \in C_0^\infty(\mathbb{R})$ and $g \in X'(\mathbb{R}, w^{-1}) \subset L_{\text{loc}}^1(\mathbb{R})$. Taking into account that T is of weak type $(1, 1)$, we see that Tf satisfies (3.3). From Theorem 3.3 we get that there exist constants $\lambda \in (0, 1)$ and $C_L > 0$ independent of f and g such that

$$\int_{\mathbb{R}} |(Tf)(x)g(x)| dx \leq C_L \int_{\mathbb{R}} M_\lambda^\#(Tf)(x) Mg(x) dx. \quad (3.5)$$

Since T is bounded on some standard Lebesgue space $L^p(\mathbb{R})$ for $1 < p < \infty$ and $L^s(J) \subset L^r(J)$ whenever $0 < r < s < \infty$ and J is a finite interval, we see that $Tf \in L_{\text{loc}}^\delta(\mathbb{R})$ for each $\delta \in (0, p]$. From Lemma 3.4 and hypothesis (c) it follows that

$$M_\lambda^\#(Tf)(x) \leq (1/\lambda)^{1/\delta} (Tf)_\delta^\#(x) \leq (1/\lambda)^{1/\delta} C_\delta Mf(x) \quad (x \in \mathbb{R}) \quad (3.6)$$

for some $\delta \in (0, 1)$. Combining (3.5) and (3.6) with Hölder's inequality (see Lemma 2.2), we obtain

$$\begin{aligned} \int_{\mathbb{R}} |(Tf)(x)g(x)| dx &\leq C_1 \int_{\mathbb{R}} Mf(x) Mg(x) dx \\ &\leq C_1 \|Mf\|_{X(\mathbb{R}, w)} \|Mg\|_{X'(\mathbb{R}, w^{-1})}, \end{aligned} \quad (3.7)$$

where $C_1 := (1/\lambda)^{1/\delta} C_\delta C_L > 0$ is independent of $f \in C_0^\infty(\mathbb{R})$ and $g \in X'(\mathbb{R}, w^{-1})$. Taking into account that M is bounded on $X(\mathbb{R}, w)$ and on $X'(\mathbb{R}, w^{-1})$, from (3.7) and we get

$$\int_{\mathbb{R}} |(Tf)(x)g(x)| dx \leq C_2 \|f\|_{X(\mathbb{R}, w)} \|g\|_{X'(\mathbb{R}, w^{-1})},$$

where $C_2 := C_1 \|M\|_{\tilde{\mathcal{B}}(X(\mathbb{R}, w))} \|M\|_{\tilde{\mathcal{B}}(X'(\mathbb{R}, w^{-1}))}$. From this inequality and (2.1) we obtain

$$\begin{aligned} \|Tf\|_{X(\mathbb{R}, w)} &= \sup \left\{ \int_{\mathbb{R}} |(Tf)(x)g(x)| dx : g \in X'(\mathbb{R}, w^{-1}), \|g\|_{X'(\mathbb{R}, w^{-1})} \leq 1 \right\} \\ &\leq C_2 \|f\|_{X(\mathbb{R}, w)} \end{aligned}$$

for all $f \in C_0^\infty(\mathbb{R})$. Taking into account that $C_0^\infty(\mathbb{R})$ is dense in $X(\mathbb{R}, w)$ in view of Lemma 2.12(a), from the latter inequality it follows that T is bounded on $X(\mathbb{R}, w)$ and (3.4) holds. \square

Remark 3.7. The proof of this result without changes extends to the case of \mathbb{R}^n .

Theorem 3.8. *Let $X(\mathbb{R})$ be a Banach function space and $w : \mathbb{R} \rightarrow [0, \infty]$ be a weight such that $w \in X_{\text{loc}}(\mathbb{R})$ and $1/w \in X'_{\text{loc}}(\mathbb{R})$. Suppose the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R}, w)$ and on $X'(\mathbb{R}, w^{-1})$.*

- (a) *If the space $X(\mathbb{R})$ is separable, then the Cauchy singular integral operator S is bounded on the space $X(\mathbb{R}, w)$ and $S^2 = I$.*
- (b) *If the space $X(\mathbb{R})$ is reflexive, then the Cauchy singular integral operator S is bounded on the spaces $X(\mathbb{R}, w)$ and $X'(\mathbb{R}, w^{-1})$ and its adjoint S^* coincides with S on the space $X'(\mathbb{R}, w^{-1})$.*

Proof. From Theorems 3.1 and 3.5 it follows that all hypotheses of Theorem 3.6 are fulfilled. Hence, the operator S is bounded on $X(\mathbb{R}, w)$.

Let now $\varphi \in X(\mathbb{R}, w)$. Then there exists a sequence $f_n \in L_0^\infty(\mathbb{R})$ such that $f_n \rightarrow \varphi$ in $X(\mathbb{R}, w)$ as $n \rightarrow \infty$. From (3.1) we get $S^2 f_n = f_n$ because $L_0^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$. Hence

$$\begin{aligned} \|S^2 \varphi - \varphi\|_{X(\mathbb{R}, w)} &\leq \|S^2 \varphi - f_n\|_{X(\mathbb{R}, w)} + \|f_n - \varphi\|_{X(\mathbb{R}, w)} \\ &= \|S^2(\varphi - f_n)\|_{X(\mathbb{R}, w)} + \|\varphi - f_n\|_{X(\mathbb{R}, w)} \\ &\leq (\|S^2\|_{\mathcal{B}(X(\mathbb{R}, w))} + 1) \|\varphi - f_n\|_{X(\mathbb{R}, w)} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Thus $S^2 \varphi = \varphi$. Part (a) is proved.

(b) From (3.2) it follows that

$$\int_{\mathbb{R}} (Sf)(x) \overline{g(x)} dx = \int_{\mathbb{R}} f(x) \overline{(Sg)(x)} dx.$$

for all $f, g \in L_0^\infty(\mathbb{R})$. From this equality and Lemmas 2.8 and 2.12(b) it follows that S is a self-adjoint and densely defined operator on $X(\mathbb{R}, w)$ and $X'(\mathbb{R}, w^{-1})$. By the standard argument (see [10, Chap. III, Section 5.5]), one can show that $S = S^* \in \mathcal{B}(X'(\mathbb{R}, w^{-1}))$ because $S \in \mathcal{B}(X(\mathbb{R}, w))$ by part (a). \square

3.4. Necessary condition

Let $X(\mathbb{R})$ be a Banach function space and $X'(\mathbb{R})$ be its associate space. We say that a weight $w: \mathbb{R} \rightarrow [0, \infty]$ belongs to the class $A_X(\mathbb{R})$ if

$$\sup_{-\infty < a < b < \infty} \frac{1}{b-a} \|w\chi_{(a,b)}\|_{X(\mathbb{R})} \|w^{-1}\chi_{(a,b)}\|_{X'(\mathbb{R})} < \infty.$$

Theorem 3.9. *Let $X(\mathbb{R})$ be a Banach function space and $w: \mathbb{R} \rightarrow [0, \infty]$ be a weight. If the operator S is bounded on the space $X(\mathbb{R}, w)$, then*

- (a) $w \in X_{\text{loc}}(\mathbb{R})$ and $1/w \in X'_{\text{loc}}(\mathbb{R})$;
- (b) $X(\mathbb{R}, w)$ is a Banach function space;
- (c) $w \in A_X(\mathbb{R})$.

Proof. (a) The idea of the proof is borrowed from [7, Lemma 3.3]. Let $E \subset \mathbb{R}$ be a measurable set of finite measure. Then there exist $a, b \in \mathbb{R}$ such that $E \subset (a, b) =: J$. It is clear that

$$w(x)\chi_E(x) \leq w(x)\chi_J(x), \quad \chi_E(x)/w(x) \leq \chi_J(x)/w(x)$$

for almost all $x \in \mathbb{R}$. Then by Axiom (A2),

$$\|w\chi_E\|_{X(\mathbb{R})} \leq \|w\chi_J\|_{X(\mathbb{R})}, \quad \|\chi_E/w\|_{X'(\mathbb{R})} \leq \|\chi_J/w\|_{X'(\mathbb{R})}.$$

Thus, it is sufficient to prove that $w\chi_J \in X(\mathbb{R})$ and $\chi_J/w \in X'(\mathbb{R})$.

Obviously, the operator $(Vf)(x) = \chi_J(x)xf(x)$ is bounded on $X(\mathbb{R}, w)$ and

$$((SV - VS)f)(x) = \frac{1}{\pi i} \int_J f(y) dy$$

for almost all $x \in \mathbb{R}$. Since the operator $SV - VS$ is bounded on $X(\mathbb{R}, w)$, there exists a constant $C_1 > 0$ such that

$$\left\| \frac{1}{\pi i} \int_J f(y) dy \right\|_{X(\mathbb{R}, w)} \leq C_1 \|f\|_{X(\mathbb{R}, w)} \quad \text{for all } f \in X(\mathbb{R}, w). \quad (3.8)$$

On the other hand,

$$\left\| \frac{1}{\pi i} \int_J f(y) dy \right\|_{X(\mathbb{R}, w)} = \frac{1}{\pi} \left| \int_J f(y) dy \right| \|w\chi_J\|_{X(\mathbb{R})}. \quad (3.9)$$

Since $w(x) > 0$ a.e. on \mathbb{R} , we have $\|w\chi_J\|_{X(\mathbb{R})} > 0$. Hence, from (3.8)–(3.9) it follows that

$$\left| \int_J f(y) dy \right| \leq \frac{C_1 \pi}{\|w\chi_J\|_{X(\mathbb{R})}} \|f\|_{X(\mathbb{R}, w)}.$$

Therefore,

$$\left| \int_{\mathbb{R}} f(y)w(y) \cdot \frac{\chi_J(y)}{w(y)} dy \right| \leq \frac{C_1 \pi}{\|w\chi_J\|_{X(\mathbb{R})}} \|fw\|_{X(\mathbb{R})}$$

for all measurable functions f such that $fw \in X(\mathbb{R})$. By Lemma 2.3, we have $\chi_J/w \in X'(\mathbb{R})$.

Let us show that there exists a function $g_0 \in X(\mathbb{R})$ such that

$$C_2 := \frac{1}{\pi} \left| \int_J \frac{g_0(y)}{w(y)} dy \right| > 0. \quad (3.10)$$

Assume the contrary. Then, taking into account Lemma 2.10(a), we obtain

$$\int_J \frac{g(y)}{w(y)} dy = 0 \quad (3.11)$$

for all g continuous on \overline{J} . By Axiom (A5), $(1/w)|_J \in L^1(J)$. Without loss of generality, assume that $|J| = 2\pi$. Let $\eta : [0, 2\pi] \rightarrow \overline{J}$ be a homeomorphism such that $|\eta'(x)| = 1$ for almost all $x \in [0, 2\pi]$. From (3.11) we get

$$\int_0^{2\pi} \frac{\varphi(x)}{w(\eta(x))} dx = 0 \quad \text{for all } \varphi \in C[0, 2\pi]. \quad (3.12)$$

Taking $\varphi(x) = e^{inx}$ with $n \in \mathbb{Z}$, we see from (3.12) that all Fourier coefficients of $1/(w \circ \eta)$ vanish. This implies that $1/w(\eta(x)) = 0$ for almost all $x \in [0, 2\pi]$. Consequently, $w(y) = \infty$ almost everywhere on J . This contradicts the assumption that w is a weight. Thus, $C_2 > 0$.

Clearly, $f_0 = g_0/w \in X(\Gamma, w)$. Then from (3.8)–(3.10) it follows that

$$\|w\chi_J\|_{X(\mathbb{R})} \leq \frac{C_1\pi}{C_2} \|f_0\|_{X(\mathbb{R}, w)},$$

that is, $w\chi_J \in X(\mathbb{R})$. Part (a) is proved.

Part (b) follows from part (a) and Lemma 2.4(b).

(c) The idea of the proof is borrowed from [7, Theorem 3.2]. By part(b), $X(\mathbb{R}, w)$ is a Banach function space.

Let Q be an arbitrary interval and Q_1, Q_2 be its two halves. Take a function $f \geq 0$ supported in Q_1 . Then for $\tau \in Q_1$ and $x \in Q_2$ we have $|\tau - x| \leq |Q|$. Therefore,

$$\begin{aligned} |(Sf)(x)| &= \frac{1}{\pi} \left| \int_{Q_1} \frac{f(\tau)}{\tau - x} d\tau \right| = \frac{1}{\pi} \int_{Q_1} \frac{f(\tau)}{|\tau - x|} d\tau \\ &\geq \frac{1}{\pi|Q|} \int_{Q_1} f(\tau) d\tau = \frac{1}{2\pi|Q_1|} \int_{Q_1} f(\tau) d\tau. \end{aligned}$$

Thus,

$$|(Sf)(x)|\chi_{Q_2}(x) \geq \frac{1}{2\pi|Q_1|} \left(\int_{Q_1} f(\tau) d\tau \right) \chi_{Q_2}(x) \quad (x \in \mathbb{R}).$$

Then, by Axioms (A1) and (A2),

$$\|Sf\|_{X(\mathbb{R}, w)} \geq \|(Sf)\chi_{Q_2}\|_{X(\mathbb{R}, w)} \geq \frac{1}{2\pi|Q_1|} \left(\int_{Q_1} f(\tau) d\tau \right) \|\chi_{Q_2}\|_{X(\mathbb{R}, w)}. \quad (3.13)$$

On the other hand, since S is bounded on $X(\mathbb{R}, w)$, we get

$$\|Sf\|_{X(\mathbb{R}, w)} \leq \|S\|_{\mathcal{B}(X(\mathbb{R}, w))} \|f\|_{X(\mathbb{R}, w)} = \|S\|_{\mathcal{B}(X(\mathbb{R}, w))} \|f\chi_{Q_1}\|_{X(\mathbb{R}, w)}. \quad (3.14)$$

Combining (3.13) and (3.14), we arrive at

$$\frac{1}{|Q_1|} \left(\int_{Q_1} f(\tau) d\tau \right) \|w\chi_{Q_2}\|_{X(\mathbb{R})} \leq 2\pi \|S\|_{\mathcal{B}(X(\mathbb{R},w))} \|f\chi_{Q_1}\|_{X(\mathbb{R},w)}. \quad (3.15)$$

Taking $f = \chi_{Q_1}$, from (3.15) we get

$$\|w\chi_{Q_2}\|_{X(\mathbb{R})} \leq 2\pi \|S\|_{\mathcal{B}(X(\mathbb{R},w))} \|w\chi_{Q_1}\|_{X(\mathbb{R})}.$$

Analogously one can obtain

$$\|w\chi_{Q_1}\|_{X(\mathbb{R})} \leq 2\pi \|S\|_{\mathcal{B}(X(\mathbb{R},w))} \|w\chi_{Q_2}\|_{X(\mathbb{R})}. \quad (3.16)$$

From (3.15) and (3.16) it follows that

$$\frac{1}{|Q_1|} \left(\int_{Q_1} f(\tau) d\tau \right) \|w\chi_{Q_1}\|_{X(\mathbb{R})} \leq C \|f\chi_{Q_1}\|_{X(\mathbb{R},w)}, \quad (3.17)$$

where $C := (2\pi \|S\|_{\mathcal{B}(X(\mathbb{R},w))})^2$. Let

$$Y := \{g \in X(\mathbb{R}, w) : \|g\|_{X(\mathbb{R},w)} \leq 1\}.$$

If $g \in Y$, then $|g|\chi_{Q_1} \geq 0$ is supported in Q_1 . Then from (3.17) we obtain

$$\|w\chi_{Q_1}\|_{X(\mathbb{R})} \int_{\mathbb{R}} |g(\tau)|\chi_{Q_1}(\tau) d\tau \leq C|Q_1| \quad (3.18)$$

for all $g \in Y$. From (2.2) we get

$$\|w^{-1}\chi_{Q_1}\|_{X'(\mathbb{R})} = \|\chi_{Q_1}\|_{X'(\mathbb{R},w^{-1})} = \sup_{g \in Y} \int_{\mathbb{R}} |g(\tau)|\chi_{Q_1}(\tau) d\tau. \quad (3.19)$$

From (3.18) and (3.19) it follows that

$$\|w\chi_{Q_1}\|_{X(\mathbb{R})} \|w^{-1}\chi_{Q_1}\|_{X'(\mathbb{R})} \leq C|Q_1|.$$

Since $Q_1 \subset \mathbb{R}$ is an arbitrary interval, we conclude that $w \in A_X(\mathbb{R})$. \square

3.5. The case of weighted variable Lebesgue spaces

We start this subsection with the following well-known fact.

Theorem 3.10 ([5, Theorems 3.2.13 and 3.4.7]). *Let $p : \mathbb{R} \rightarrow [1, \infty]$ be a measurable a.e. finite function satisfying (1.1). Then $L^{p(\cdot)}(\mathbb{R})$ is a separable and reflexive Banach function space whose associate space is isomorphic to $L^{p'(\cdot)}(\mathbb{R})$.*

Now we are in a position to give a proof of Theorem 1.3.

Proof of Theorem 1.3. Necessity. Theorem 3.10 immediately implies that if p satisfies (1.1), then $L^{p(\cdot)}(\mathbb{R})$ is a Banach function space and

$$\mathcal{A}_{p(\cdot)}(\mathbb{R}) = A_{L^{p(\cdot)}}(\mathbb{R}).$$

From Theorem 3.9 it follows that that if S is bounded on the space $L^{p(\cdot)}(\mathbb{R}, w)$, then $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R})$. The necessity portion is proved.

Sufficiency. From Theorem 3.10 we know that $L^{p(\cdot)}(\mathbb{R})$ is a separable and reflexive Banach function space. If $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R})$, then $w \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R})$, $1/w \in L_{\text{loc}}^{p'(\cdot)}(\mathbb{R})$,

and $1/w \in \mathcal{A}_{p'(\cdot)}(\mathbb{R})$. Further, it is easy to see that p is globally log-Hölder continuous if and only if so is p' . Hence, by Theorem 1.2, the Hardy-Littlewood maximal function is bounded on $L^{p(\cdot)}(\mathbb{R}, w)$ and on $L^{p'(\cdot)}(\mathbb{R}, w^{-1})$. Applying Theorem 3.8(a), we see that the operator S is bounded on $L^{p(\cdot)}(\mathbb{R}, w)$. This finishes the proof of Theorem 1.3. \square

Theorem 1.4 follows immediately from Theorems 1.2, 3.8, and 3.10.

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